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Anomalous localized resonance on smooth domains using spectral properties of the Neumann-Poincaré operators

Kazunori Ando

Department of Electrical ane Electronic Engineering and Computer Science,
Graduate School of Science and Engineering, Ehime University

Joint work with Hyeonbae Kang
Department of Mathematics, Inha University, Republic of Korea

1 Introduction

Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^d$, $d=2,3$. We assume that Ω is occupied with a material which has the dielectric constant $\epsilon_c<0$ with dissipation $\delta>0$ and the matrix $\mathbb{R}^d\setminus\overline{\Omega}$ has the dielectric constant $\epsilon_m>0$. So the total distribution of the dielectric constant is written as

$$\epsilon=\begin{cases}\epsilon_c+i\delta, & \text{in } \Omega,\\ \epsilon_m, & \text{in } \mathbb{R}\setminus\overline{\Omega},\end{cases}$$

which is called the plasmonic structure.

We also assume that Ω is diametrically small and there exists the polarizable dipole source $a\cdot\nabla\delta_z$ outside Ω, where $a\in\mathbb{R}^d$ is a constant vector and δ_z is the Dirac mass at $z\in\mathbb{R}^d\setminus\overline{\Omega}$. Then we consider the following dielectric equation under the quasi-static approximation:

$$\begin{cases}\nabla\cdot\epsilon\nabla u=a\cdot\nabla\delta_z & \text{in } \mathbb{R}^d,\\ u(x)=O(|x|^{1-d}) & \text{as } |x|\to\infty.\end{cases} \tag{1}$$

Let u_δ be the solution of (1). The resonance is characterized by the blow-up of $\|\nabla u_\delta\|_{L^2(\Omega)}$:

$$\|\nabla u_\delta\|_{L^2(\Omega)}\to\infty \quad \text{as } \delta\to 0. \tag{2}$$

In particular, we are interested in anomalous localized resonance, which is characterized as follows:

1. $E_\delta:=\delta\|\nabla u_\delta\|^2_{L^2(\Omega)}\to\infty$ as $\delta\to 0$,

2. there exist $R > 0$ and $C > 0$ such that $|u_\delta(x)| < C$ for $|x| > R$.

Anomalous localized resonance (ALR) is discovered in [13], and applied to cloaking by anomalous localized resonance (CALR) [11]. There are many results on this subject; see e.g. [1] and the references therein. So far, however, ALR has been mainly studied in the core-shell structure.

Here we consider simply connected structure for Ω, and show that ALR occurs on ellipse in two dimensions; on the other hand, it does not occur on ball in three dimensions. We emphasize that in [12] the authors consider the plasmonic structure on disk in two dimensions and showed that the complete resonance occurs.

2 Neumann-Poincaré operator and symmetrization

Let Γ be the fundamental solution to the Laplacian on $\mathbb{R}^d$, $d = 2, 3$, which is given by

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x|, & d = 2, \\ -\dfrac{1}{4\pi} |x|^{-1}, & d = 3. \end{cases}$$

The single layer potential on $\partial\Omega$ is defined by

$$\mathcal{S}_{\partial\Omega}[\varphi](x) = \int_{\partial\Omega} \Gamma(x-y)\varphi(y) \mathrm{d}\sigma(y), \quad x \in \mathbb{R}^d.$$

and the Neumann-Poincaré (NP) operator by

$$\mathcal{K}_{\partial\Omega}[\varphi](x) = \int_{\partial\Omega} \frac{\partial}{\partial \nu_y} \Gamma(x-y)\varphi(y) \mathrm{d}\sigma(y), \quad x \in \partial\Omega,$$

whose $L^2(\partial\Omega)$-adjoint $\mathcal{K}^*_{\partial\Omega}$ is also called the NP operator. Here we denote by $\frac{\partial}{\partial \nu_y}$ the outward normal derivative in y-variable on $\partial\Omega$. There holds the following jump relation:

$$\frac{\partial}{\partial \nu} \mathcal{S}_{\partial\Omega}[\varphi] \Big|_{\pm} (x) = \left(\pm\frac{1}{2} + \mathcal{K}^*_{\partial\Omega} \right) [\varphi](x), \quad x \in \partial\Omega, \tag{3}$$

where the subscript $+$ (resp. $-$) indicate the limits to $\partial\Omega$ from outside (resp. inside) of Ω. Moreover, the Plemelj's symmetrization principle (also known as Calderón's identity) holds:

$$\mathcal{S}_{\partial\Omega} \mathcal{K}^*_{\partial\Omega} = \mathcal{K}_{\partial\Omega} \mathcal{S}_{\partial\Omega}. \tag{4}$$

We denote $H^s(\partial\Omega)$, $s \in \mathbb{R}$, the L^2-Sobolev space on $\partial\Omega$, whose norm is written as $\| \cdot \|_s$. Define

$$\langle \varphi, \psi \rangle_{\mathcal{H}^*} := -\langle \varphi, \mathcal{S}_{\partial\Omega}[\psi] \rangle_{L^2(\partial\Omega)} \tag{5}$$

for $\varphi, \psi \in H_0^{-1/2}(\partial\Omega) := \{\varphi \in H^{1/2}(\partial\Omega); \langle \varphi, 1\rangle_{L^2(\partial\Omega)} = 0\}$. Note that the right hand side of (5) is well-defined, since $\mathcal{S}_{\partial\Omega}$ maps $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$. It is known that $\langle\cdot,\cdot\rangle_{\mathcal{H}^*}$ is an inner product on $H_0^{-1/2}(\partial\Omega)$, which induces the norm equivalent to the original norm of $H^{-1/2}(\partial\Omega)$:

$$\|\varphi\|_{\mathcal{H}^*} \approx \|\varphi\|_{-1/2} \tag{6}$$

for all $\varphi \in H_0^{-1/2}(\partial\Omega)$; see [8]. Put $\mathcal{H}_0^* := H_0^{-1/2}(\partial\Omega)$ equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}^*}$. Then the Plemelj's symmetrization principle (4) implies that $\mathcal{K}_{\partial\Omega}^*$ is self-adjoint on $\mathcal{H}_0^*$.

Let us consider the symmetrization of $\mathcal{K}_{\partial\Omega}$. If $\mathcal{S}_{\partial\Omega}$ is invertible, multiplying $\mathcal{S}_{\partial\Omega}^{-1}$ from the right and left side in (4), we can obtain an analogue of the Plemelj's symmetrization for $\mathcal{K}_{\partial\Omega}$ and symmtrize it in the same way as in the case of $\mathcal{K}_{\partial\Omega}^*$. It is true in three dimensions; unfortunately, there exists a domain Ω in two dimensions such that $\mathcal{S}_{\partial\Omega}[\varphi_0] = 0$ in Ω for a nontrivial $\varphi_0 \in H^{-1/2}(\partial\Omega)$, see [14].

To overcome this difficulty, we define a variant of the single layer potential

$$\tilde{\mathcal{S}}_{\partial\Omega}[\varphi] = \begin{cases} \mathcal{S}_{\partial\Omega}[\varphi], & \text{if } \langle \varphi, 1\rangle_{L^2(\partial\Omega)} = 0, \\ -1, & \text{if } \varphi = \varphi_0, \end{cases}$$

where $\varphi_0 \in H^{-1/2}$ is an eigenfunction of $\mathcal{K}_{\partial\Omega}^*$ corresponding to the eigenvalue $1/2$ which is normalized as

$$\langle \varphi_0, 1\rangle_{L^2(\partial\Omega)} = 1. \tag{7}$$

Here we note that $\mathcal{K}_{\partial\Omega}^*$ is compact on $H^{-1/2}(\partial\Omega)$ and its spectrum $\sigma(\mathcal{K}_{\partial\Omega}^*) \subset (-1/2, 1/2]$; moreover, $1/2$ is simple, since $\sigma(\mathcal{K}_{\partial\Omega}^*|_{H_0^{-1/2}(\partial\Omega)}) \subset (-1/2, 1/2)$ and $\dim H^{1/2}(\partial\Omega) \backslash H_0^{-1/2}(\partial\Omega) = 1$ (see [3, 8, 10]). Then we have an extention of (4):

$$\tilde{\mathcal{S}}_{\partial\Omega}\mathcal{K}_{\partial\Omega}^* = \mathcal{K}_{\partial\Omega}\tilde{\mathcal{S}}_{\partial\Omega},$$

and extend

$$\langle \varphi, \psi\rangle_{\mathcal{H}^*} = -\langle \varphi, \tilde{\mathcal{S}}_{\partial\Omega}[\psi]\rangle_{L^2(\partial\Omega)} \tag{8}$$

for $\varphi, \psi \in H^{-1/2}(\partial\Omega)$. Note that (6) is also extended to $H^{-1/2}(\partial\Omega)$. Then (8) is an inner product on $H^{-1/2}(\partial\Omega)$. We define the Hilbert space $\mathcal{H}^* = H^{-1/2}(\partial\Omega)$ equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}^*}$, on which $\mathcal{K}_{\partial\Omega}^*$ is symmetrized.

Let us symmetrize $\mathcal{K}_{\partial\Omega}$. $\tilde{\mathcal{S}}_{\partial\Omega}$ is a bijection from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$, so an analogue of the Plemelj's symmetrization holds:

$$\tilde{\mathcal{S}}_{\partial\Omega}^{-1}\mathcal{K}_{\partial\Omega} = \mathcal{K}_{\partial\Omega}^*\tilde{\mathcal{S}}_{\partial\Omega}^{-1}. \tag{9}$$

Define

$$\langle f, g\rangle_{\mathcal{H}} := -\langle f, \tilde{\mathcal{S}}_{\partial\Omega}^{-1}[g]\rangle_{L^2(\partial\Omega)} \tag{10}$$

for $f, g \in H^{1/2}(\partial\Omega)$. Then (10) is an inner product on $H^{1/2}(\partial\Omega)$ which induces the equivalence $\|\cdot\|_{\mathcal{H}} \approx \|\cdot\|_{1/2}$. Put $\mathcal{H} = H^{1/2}(\partial\Omega)$ equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Then $\mathcal{K}_{\partial\Omega}$ is symmetrized on $\mathcal{H}$ by (9). We note that $\tilde{S}_{\partial\Omega}$ is unitaty and $\{\tilde{S}_{\partial\Omega}[\psi_j]\}_{j=1}^{\infty} \cup \{-1\}$ is an orthonormal bases of $\mathcal{H}$, where $\{\psi_j\}$ is an orthonormal basis of $\mathcal{H}_0^*$. In particular, we can choose $\{\psi_j\}_{j=1}^{\infty}$ as the normalized eigenvectors of $\mathcal{K}_{\partial\Omega}^*$ on $\mathcal{H}_0^*$.

For the details of the symmetrization of the NP operator, see [5].

3 Representation of the solution

If $\partial\Omega$ is smooth, at least $C^{1,\alpha}$ for some $\alpha > 0$, then it is known that $\mathcal{K}_{\partial\Omega}^*$ is compact on $\mathcal{H}^*$ (see [9], also [10]). Since $\mathcal{K}_{\partial\Omega}^*$ is self-adjoint on $\mathcal{H}_0^*$, its eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ accumulate to 0. We remark that $|\lambda_j| < 1/2$ (see [6, 14]). Let ψ_j be the normalized eigenfunction corresponding to the eigenvalue λ_j in $\mathcal{H}_0^*$. Then we have an orthonormal basis $\tilde{S}_{\partial\Omega}[\{\psi_j\}_{j=0}^{\infty} \cup \{\varphi_0\}] = \{\mathcal{S}_{\partial\Omega}[\psi_j]\}_{j=1}^{\infty} \cup \{-1\}$ in $\mathcal{H}$.

Fix $z \in \mathbb{R}^d \setminus \overline{\Omega}$. Then $\Gamma(\cdot - z)$ belongs to $H^{1/2}(\partial\Omega)$, and so admits the following expansion:

$$\Gamma(x-z) = \sum_{j=1}^{\infty} c_j(z)\mathcal{S}_{\partial\Omega}[\psi_j](x) + c_0(z), \quad x \in \partial\Omega, \tag{11}$$

for some constants $c_j(z)$ (depending on z) which satisfy

$$\sum_{j=1}^{\infty} |c_j(z)|^2 < \infty.$$

Since $-\langle \mathcal{S}_{\partial\Omega}[\psi_i], \psi_j\rangle_{L^2(\partial\Omega)} = \delta_{ij}$, where δ_{ij} is the Kronecker's delta, we see that

$$c_j(z) = -\int_{\partial\Omega} \Gamma(x-z)\psi_j(x)\mathrm{d}\sigma(x) = -\mathcal{S}_{\partial\Omega}[\psi_j](z), \quad j = 1, 2, 3, \ldots.$$

We also see from (7) that

$$c_0(z) = \mathcal{S}_{\partial\Omega}[\varphi_0](z).$$

So, we obtain the following formula:

$$\Gamma(x-z) = -\sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z)\mathcal{S}_{\partial\Omega}[\psi_j](x) + \mathcal{S}_{\partial\Omega}[\varphi_0](z), \quad x \in \partial\Omega. \tag{12}$$

Observe that

$$\left\| \sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z)\mathcal{S}_{\partial\Omega}[\psi_j] \right\|_{\mathcal{H}}^2 = \sum_{j=1}^{\infty} |\mathcal{S}_{\partial\Omega}[\psi_j](z)|^2 < \infty. \tag{13}$$

Since $\|\cdot\|_{\mathcal{H}} \approx \|\cdot\|_{1/2}$, we find from the trace theorem that $\sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z)\mathcal{S}_{\partial\Omega}[\psi_j]$ converges in $H^1(\Omega)$ and harmonic in Ω. So, we obtain the following expansion formula of the fundamental solution to the Laplacian.

Theorem 3.1. *It holds that*

$$\Gamma(x-z) = -\sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z)\mathcal{S}_{\partial\Omega}[\psi_j](x) + \mathcal{S}_{\partial\Omega}[\varphi_0](z), \quad x \in \overline{\Omega}, z \in \mathbb{R}^d \setminus \overline{\Omega}. \tag{14}$$

We now derive a representation of the solution to (1). By the jump relation (3), the equation (1) is equivalent to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \Delta u = a \cdot \nabla \delta_z & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ u|_- = u|_+, \quad (\epsilon_c + i\delta) \left.\dfrac{\partial u}{\partial \nu}\right|_- = \epsilon_m \left.\dfrac{\partial u}{\partial \nu}\right|_+ & \text{on } \partial\Omega, \\ u(x) = O(|x|^{1-d}) & \text{as } |x| \to \infty. \end{cases} \tag{15}$$

We seek the solution of (15) in the following form:

$$u_\delta(x) = F_z(x) + \mathcal{S}_{\partial\Omega}[\varphi_\delta](x), \quad x \in \mathbb{R}^d. \tag{16}$$

where the potential $\varphi_\delta \in \mathcal{H}_0^*$ and

$$F_z(x) := a \cdot \nabla_x \Gamma(x-z), \quad x \neq z. \tag{17}$$

Note that $\Delta F_z(x) = a \cdot \nabla \delta_z(x)$. The solution $u_\delta(x)$ satisfies the equation of (15) on Ω and $\mathbb{R}^d \setminus \overline{\Omega}$; moreover, $u_\delta(x)$ decays as $O(|x|^{1-d})$, since $\varphi_0 \in \mathcal{H}_0^*$. Then, from the transmission condision in (15), we should solve the following integral equation

$$(\lambda_\delta I - \mathcal{K}_{\partial\Omega}^*)[\varphi_\delta] = \partial_\nu F_z \quad \text{on } \partial\Omega \tag{18}$$

($\partial_\nu F_z$ denotes the outward normal derivative of F_z on $\partial\Omega$). Here

$$\lambda_\delta := \frac{\epsilon_c + \epsilon_m + i\delta}{2(\epsilon_c - \epsilon_m) + 2i\delta} \to \frac{\epsilon_c + \epsilon_m}{2(\epsilon_c - \epsilon_m)} \quad \text{as } \delta \to 0.$$

From the spectral resolution of $\mathcal{K}_{\partial\Omega}^*$ on $\mathcal{H}_0^*$

$$\mathcal{K}_{\partial\Omega}^* = \sum_{j=1}^{\infty} \lambda_j \psi_j \otimes \psi_j, \tag{19}$$

the solution φ_δ to the integral equation (18) can be representad as

$$\varphi_\delta = \sum_{j=1}^{\infty} \frac{\alpha_j(z)}{\lambda_\delta - \lambda_j} \psi_j, \tag{20}$$

where

$$\alpha_j(z) := \langle \partial_\nu F_z, \psi_j \rangle_{\mathcal{H}^*}.$$

We can see from (17) that

$$\alpha_j(z) = -a \cdot \nabla \int_{\partial\Omega} \frac{\partial}{\partial \nu_x} \Gamma(x-z) \mathcal{S}_{\partial\Omega}[\psi_j](x) \mathrm{d}\sigma(x).$$

From (3) and (14), we have

$$\frac{\partial}{\partial \nu_x} \Gamma(x-z) = -\sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z) \frac{\partial}{\partial \nu} \mathcal{S}_{\partial\Omega}[\psi_j](x) = \sum_{j=1}^{\infty} \left(\frac{1}{2} - \lambda_j\right) \mathcal{S}_{\partial\Omega}[\psi_j](z) \psi_j(x).$$

It then follows that

$$\alpha_j(z) = \left(\frac{1}{2} - \lambda_j\right) a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_j](z). \tag{21}$$

4 Anomalous localized resonance

The resonance occurs if and only if $\frac{\epsilon_c + \epsilon_m}{2(\epsilon_c - \epsilon_m)} \in \sigma(\mathcal{K}^*_{\partial\Omega})$; if $\frac{\epsilon_c + \epsilon_m}{2(\epsilon_c - \epsilon_m)}$ is a non-zero eigenvalue of the NP operator, resonance occurs in the sense of (2) and have asymptotics $\|\nabla u_\delta\|_{L^2(\Omega)} \sim \delta$ as $\delta \to 0$; however, it is not localized, see [5].

Let us consider the resonance at the accumulation point of $\sigma(\mathcal{K}^*_{\partial\Omega})$, *i.e.*, $\epsilon_c + \epsilon_m = 0$. We assume that 0 is not an eigenvalue of $\mathcal{K}^*_{\partial\Omega}$. Since $\partial\Omega$ is smooth, $\mathcal{K}^*_{\partial\Omega}$ is compact, hence 0 is an essential spectrum. It is worth mentioning that we are not aware of any domain other than disks on which the NP operator has 0 as an eigenvalue. If Ω is a disk, then $\mathcal{K}^*_{\partial\Omega} \equiv 0$ on $\mathcal{H}^*_0$.

When $\epsilon_c + \epsilon_m = 0$, we have

$$\lambda_\delta \approx \delta.$$

We first show that

$$\|\mathcal{S}_{\partial\Omega}[\varphi]\|^2_{L^2(\Omega)} \approx \|\varphi\|^2_{\mathcal{H}^*} \tag{22}$$

for all $\varphi \in \mathcal{H}^*_0$. In fact, we see from (3) and (19) that

$$\begin{aligned}
\|\mathcal{S}_{\partial\Omega}[\varphi]\|^2_{L^2(\Omega)} &= \int_{\partial\Omega} \mathcal{S}_{\partial\Omega}[\varphi] \overline{\frac{\partial}{\partial \nu} \mathcal{S}_{\partial\Omega}[\varphi]\Big|_-} \mathrm{d}\sigma \\
&= \left\langle \varphi, \left(-\frac{1}{2}I + \mathcal{K}^*_{\partial\Omega}\right)[\varphi] \right\rangle_{\mathcal{H}^*} \\
&= \sum_{j=1}^{\infty} \left(\frac{1}{2} - \lambda_j\right) |\langle \varphi, \psi_j \rangle_{\mathcal{H}^*}|^2.
\end{aligned}$$

Since $|\lambda_j| < 1/2$ and accumulates to 0, we have (22). Then we see from (20)

$$\|\nabla(u_\delta - F_z)\|_{L^2(\Omega)} \approx \|\varphi_\delta\|^2_{\mathcal{H}^*} \equiv \sum_{j=1}^{\infty} \frac{|\alpha_j(z)|^2}{\delta^2 + \lambda_j^2}.$$

4.1 Anomalous localized resonance on ellipse and cloaking on ellipses

Assume that Ω is an ellipse in $\mathbb{R}^2$. We use the elliptic coordinates

$$x = (x_1, x_2) = (x_1(\rho,\theta), x_2(\rho,\theta)) \in \mathbb{R}^2,$$

where

$$x_1(\rho,\theta) = R\cos\theta\cosh\rho, \quad x_2(\rho,\theta) = R\sin\theta\sinh\rho, \quad \rho > 0, 0 \le \theta < 2\pi.$$

For $\rho_0 > 0$, let

$$\partial\Omega = \{(x_1(\rho_0,\omega), x_2(\rho_0,\omega)) \in \mathbb{R}^2;\ 0 \le \omega < 2\pi\}. \tag{23}$$

Then $\partial\Omega$ is an ellipse whose foci are $(\pm R, 0)$. The length element $d\sigma$ and the outward normal derivative $\frac{\partial}{\partial\nu}$ are given in terms of the elliptic coordinates by

$$d\sigma = \Xi d\omega, \quad \frac{\partial}{\partial\nu} = \Xi^{-1}\frac{\partial}{\partial\rho},$$

where

$$\Xi = \Xi(\rho_0,\omega) := R\sqrt{\sinh^2\rho_0 + \sin^2\omega}.$$

Let us define

$$\phi_n^c(\omega) := \Xi(\rho_0,\omega)^{-1}\cos n\omega, \quad \phi_n^s(\omega) := \Xi(\rho_0,\omega)^{-1}\sin n\omega, \quad n = 1,2,\cdots.$$

Then we have

$$\mathcal{K}_{\partial\Omega}^*[\phi_n^c](\omega) = \alpha_n\phi_n^c(\omega), \quad \mathcal{K}_{\partial\Omega}^*[\phi_n^s](\omega) = -\alpha_n\phi_n^s(\omega),$$

where

$$\alpha_n = \frac{1}{2e^{2n\rho_0}}, \quad n = 1,2,\cdots.$$

$\{\cos n\omega, \sin n\omega;\ n = 1,2,\cdots\}$ is complete in $L_0^2(\partial\Omega)$, hence in $\mathcal{H}_0^*$ (see [10]), which means that $\mathcal{K}_{\partial\Omega}^*$ has the following eigenfunction expansion in $\mathcal{H}_0^*$:

$$\mathcal{K}_{\partial\Omega}^* = \sum_{n=1}^{\infty}\alpha_n\psi_n^c\otimes\psi_n^c - \sum_{n=1}^{\infty}\alpha_n\psi_n^s\otimes\psi_n^s,$$

where

$$\psi_n^c := \sqrt{\frac{ne^{n\rho_0}}{\pi\cosh n\rho_0}}\phi_n^c, \quad \psi_n^s := \sqrt{\frac{ne^{n\rho_0}}{\pi\sinh n\rho_0}}\phi_n^s. \tag{24}$$

Note that $\{\psi_n^c, \psi_n^s;\ n = 1,2,\cdots\}$ is an orthonormal basis in $\mathcal{H}_0^*$. We also have

$$\mathcal{S}_{\partial\Omega}[\phi_n^c](x) = \begin{cases} -\dfrac{\cosh n\rho}{ne^{n\rho_0}}\cos n\theta, & \rho \le \rho_0, \\ -\dfrac{\cosh n\rho_0}{ne^{n\rho}}\cos n\theta, & \rho > \rho_0, \end{cases} \tag{25}$$

$$\mathcal{S}_{\partial\Omega}[\phi_n^s](x) = \begin{cases} -\dfrac{\sinh n\rho}{ne^{n\rho_0}} \sin n\theta, & \rho \le \rho_0, \\ -\dfrac{\sinh n\rho_0}{ne^{n\rho}} \sin n\theta, & \rho > \rho_0, \end{cases} \tag{26}$$

where $\mathcal{S}_{\partial\Omega}[\varphi]$ is the single layer potential of φ. See [4, 7].

Furthermore, by the change of variables,

$$\frac{\partial}{\partial x_1} = \frac{1}{R(\sinh^2\rho + \sin^2\theta)} \left(\cos\theta \sinh\rho \frac{\partial}{\partial\rho} - \sin\theta\cosh\rho\frac{\partial}{\partial\theta} \right), \tag{27}$$

$$\frac{\partial}{\partial x_2} = \frac{1}{R(\sinh^2\rho + \sin^2\theta)} \left(\sin\theta \cosh\rho \frac{\partial}{\partial\rho} + \cos\theta\sinh\rho\frac{\partial}{\partial\theta} \right). \tag{28}$$

Theorem 4.1. *Suppose that Ω is an ellipse given by (23). Then we have*

$$\|\nabla u_\delta\|^2_{L^2(\Omega)} \sim \begin{cases} \delta^{-3+\rho_z/\rho_0}|\log\delta| & \text{if } \rho_0 < \rho_z < 3\rho_0, \\ |\log\delta|^2 & \text{if } \rho_z = 3\rho_0, \\ 1 & \text{if } \rho_z > 3\rho_0, \end{cases} \tag{29}$$

as $\delta \to 0$.

Therefore, the quantity $E_\delta = \delta\|u_\delta\|^2_{L^2(\Omega)}$ blows up if $\rho_0 < \rho_z \le 2\rho_o$ while it tends to 0 as $\delta \to 0$ if $\rho_z > 2\rho_0$.

Proof. We only have to study the asymptotics of the following summation:

$$\begin{aligned} &\sum_{j=1}^\infty \frac{|a\cdot\nabla\mathcal{S}_{\partial\Omega}[\psi_j](z)|^2}{\delta^2+\lambda_j^2} \\ &= \sum_{n=1}^\infty \frac{1}{\delta^2+\alpha_n^2}\,|a\cdot\nabla\mathcal{S}_{\partial\Omega}[\psi_n^c](z)|^2 + \sum_{n=1}^\infty \frac{1}{\delta^2+\alpha_n^2}\,|a\cdot\nabla\mathcal{S}_{\partial\Omega}[\psi_n^s](z)|^2 \\ &= \sum_{n=1}^\infty \frac{1}{\delta^2+\alpha_n^2}\cdot\frac{e^{n\rho_0}\cosh n\rho_0}{n\pi}\left|a\cdot\nabla\left(e^{-n\rho}\cos n\theta\right)\right|^2 \\ &\quad + \sum_{n=1}^\infty \frac{1}{\delta^2+\alpha_n^2}\cdot\frac{e^{n\rho_0}\sinh n\rho_0}{n\pi}\left|a\cdot\nabla\left(e^{-n\rho}\sin n\theta\right)\right|^2. \end{aligned} \tag{30}$$

Since $\cosh n\rho_0 \approx \sinh n\rho_0 \approx e^{n\rho_0}$,

$$\sum_{j=1}^\infty \frac{|a\cdot\nabla\mathcal{S}_{\partial\Omega}[\psi_j](z)|^2}{\delta^2+\lambda_j^2} \sim \sum_{n=1}^\infty \frac{1}{\delta^2+\lambda_n^2}\cdot\frac{e^{2n\rho_0}}{n}\left[\left|a\cdot\nabla_z\left(e^{-n\rho_z}\cos n\omega_z\right)\right|^2 + \left|a\cdot\nabla_z\left(e^{-n\rho_z}\sin n\omega_z\right)\right|^2\right]. \tag{31}$$

Let $U(\omega)$ be the rotation by the angle ω, namely,

$$U(\omega) = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix}.$$

Using the change of variable formula (27) and (28), we have

$$a \cdot \nabla \left(e^{-n\rho} \cos n\theta\right) = \frac{-ne^{n\rho}}{R\left(\sinh^2 \rho + \sin^2 \theta\right)} a \cdot U(n\theta) b(\rho, \theta), \tag{32}$$

$$a \cdot \nabla \left(e^{-n\rho} \sin n\theta\right) = \frac{-ne^{-n\rho}}{R\left(\sinh^2 \rho + \sin^2 \theta\right)} a \cdot U(n\theta - \pi/2) b(\rho, \theta), \tag{33}$$

where

$$b(\rho, \theta) = (\cos\theta \sinh\rho, \sin\theta \cosh\rho) \in \mathbb{R}^2,$$

which implies

$$\left|a \cdot \nabla \left(e^{-n\rho} \cos n\theta\right)\right|^2 + \left|a \cdot \nabla \left(e^{-n\rho} \sin n\theta\right)\right|^2 = \frac{n^2 e^{-2n\rho} |a|^2 |b(\rho, \theta)|^2}{R^2 \left(\sinh^2 \rho + \sin^2 \theta\right)^2}. \tag{34}$$

From (31) and (34), we have

$$\sum_{j=1}^{\infty} \frac{|a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_j](z)|^2}{\delta^2 + \lambda_j^2} \sim \sum_{n=1}^{\infty} \frac{n e^{2n\rho_0} e^{-2n\rho}}{\delta^2 + \frac{1}{4} e^{-4n\rho_0}}. \tag{35}$$

Let

$$N = \left[-\frac{1}{2\rho_0} \log 2\delta\right],$$

which is the first integer such that $\delta > \frac{1}{2} e^{-2N\rho_0}$. Then one can easily see that

$$\sum_{n=1}^{\infty} \frac{n e^{2n\rho_0} e^{-2n\rho}}{\delta^2 + \frac{1}{4} e^{-4n\rho_0}} = \sum_{n \le N} + \sum_{n > N} \sim \sum_{n \le N} \frac{n e^{2n\rho_0} e^{-2n\rho}}{e^{-4n\rho_0}} + \frac{1}{\delta^2} \sum_{n > N} n e^{-2n(\rho_z - \rho_0)}.$$

Observe that

$$\sum_{n \le N} \frac{n e^{2n\rho_0} e^{-2n\rho}}{e^{-4n\rho_0}} \sim \sum_{n \le N} n e^{2n(3\rho_0 - \rho_z)} \sim \begin{cases} |\log\delta| \delta^{-3+\rho_z/\rho_0} & \text{if } \rho_0 < \rho_z < 3\rho_0, \\ |\log\delta|^2 & \text{if } \rho_z = 3\rho_0, \\ 1 & \text{if } \rho_z > 3\rho_0. \end{cases}$$

On the other hand, we have

$$\frac{1}{\delta^2} \sum_{n > N} n e^{-2n(\rho_z - \rho_0)} \sim |\log\delta| \delta^{-3+\rho_z/\rho_0}.$$

So we infer that

$$\sum_{j=1}^{\infty} \frac{|a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_j](z)|^2}{\delta^2 + \lambda_j^2} \sim \begin{cases} |\log\delta| \delta^{-3+\rho_z/\rho_0} & \text{if } \rho_0 < \rho_z < 3\rho_0, \\ |\log\delta|^2 & \text{if } \rho_z = 3\rho_0, \\ 1 & \text{if } \rho_z > 3\rho_0. \end{cases}$$

Since $\|\nabla F_z\|_{L^2(\Omega)}^2$ is bounded, we obtain (29) $\square$

To show ALR, we prove the following theorem.

Theorem 4.2. *Let Ω be an ellipse given by (23). It holds for all x satisfying $\rho_x+\rho_z-4\rho_0>0$ that*

$$|u_\delta(x)-F_z(x)|\lesssim\sum_{n=1}^{\infty}e^{-n(\rho_x+\rho_z-4\rho_0)}. \tag{36}$$

In particular, let $\overline{\rho}>0$ be such that $\overline{\rho}>4\rho_0-\rho_z$, then there exists some $C=C_{\overline{\rho}}>0$ such that

$$\sup_{\rho_x\geq\overline{\rho}}|u_\delta(x)-F_z(x)|<C. \tag{37}$$

Proof. One can see from (16), (20) and (21) that

$$u_\delta(x)-F_z(x)=\sum_{n=1}^{\infty}\frac{1}{i\delta-\lambda_n}\left(\frac{1}{2}-\lambda_n\right)\{(a\cdot\nabla_z\mathcal{S}_{\partial\Omega}[\psi_n^c](z))\mathcal{S}_{\partial\Omega}[\psi_n^c](x)$$
$$+(a\cdot\nabla_z\mathcal{S}_{\partial\Omega}[\psi_n^s](z))\mathcal{S}_{\partial\Omega}[\psi_n^s](x)\}.$$

It then follows from (24), (25) and (26) that

$$\begin{aligned}u_\delta(x)-F_z(x)=&\sum_{n=1}^{\infty}\frac{1}{i\delta-\lambda_n}\left(\frac{1}{2}-\lambda_n\right)\\&\cdot\left\{\frac{e^{n\rho_0}\cosh n\rho_0}{n\pi}\left(a\cdot\nabla_z(e^{-n\rho_z}\cos n\omega_z)\right)e^{-n\rho_x}\cos n\omega_x\right.\\&\left.+\frac{e^{n\rho_0}\sinh n\rho_0}{n\pi}\left(a\cdot\nabla_z(e^{-n\rho_z}\sin n\omega_z)\right)e^{-n\rho_x}\sin n\omega_x\right\},\end{aligned}$$

where (ρ_z,ω_z) is the elliptic coordinates of z. Therefore, we have

$$|u_\delta(x)-F_z(x)|\lesssim\sum_{n=1}^{\infty}\frac{e^{4n\rho_0}}{n}\left\{|a\cdot\nabla_z(e^{-n\rho_z}\cos n\omega_z)|+|a\cdot\nabla_z(e^{-n\rho_z}\sin n\omega_z)|\right\}e^{-n\rho_x}.$$

We then see from (34) that

$$|u_\delta(x)-F_z(x)|\lesssim\sum_{n=1}^{\infty}\frac{e^{4n\rho_0}}{n}ne^{-n\rho_z}e^{-n\rho_x}=\sum_{n=1}^{\infty}e^{-n(\rho_x+\rho_z-4\rho_0)},$$

which proves (36). (37) is an immediate consequence of (36). □

Therefore, Theorems 4.1 and 4.2 imply that ALR occurs on ellipses in two dimensions.

4.2 Anomalous localized reaonance on balls

Assume that Ω is a ball in $\mathbb{R}^3$. We use the spherical coordinates

$$x=(x_1,x_2,x_3)=(x_1(r,\theta,\varphi),x_2(r,\theta,\varphi),x_3(r,\theta,\varphi))\in\mathbb{R}^3$$

where

$$x_1(r,\theta,\varphi) = r\cos\theta\sin\varphi, \quad x_2(r,\theta,\varphi) = r\sin\theta\sin\varphi, \quad x_3(r,\theta,\varphi) = r\cos\varphi,$$
$$r \geq 0, 0 \leq \theta < 2\pi, 0 \leq \varphi \leq \pi.$$

Then we have $\partial\Omega = \{x \in \mathbb{R}^3;\ |x| = r_0\}$. The surface element $d\sigma$ and the outward normal derivative $\frac{\partial}{\partial\nu}$ are given in terms of the spherical coordinates by

$$d\sigma = r_0^2 \sin\varphi d\theta d\varphi, \quad \frac{\partial}{\partial\nu} = \frac{\partial}{\partial r}.$$

The Cartesian partial derivatives in the spherical coordinates are given by

$$\frac{\partial}{\partial x_1} = \cos\theta\sin\varphi\frac{\partial}{\partial r} - \frac{\sin\theta}{r\sin\varphi}\frac{\partial}{\partial\theta} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\varphi},$$
$$\frac{\partial}{\partial x_2} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\theta}{r\sin\varphi}\frac{\partial}{\partial\theta} + \frac{\sin\theta\cos\varphi}{r}\frac{\partial}{\partial\varphi},$$
$$\frac{\partial}{\partial x_3} = \cos\varphi\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\varphi}.$$

Let $Y_n^m(\hat{x})$ be the orthonormal spherical harmonics of degree n, where $\hat{x} = \hat{x}(\theta,\varphi) = x/|x|$:

$$Y_n^m(\theta,\varphi) = (-1)^{(m+|m|)/2}\sqrt{\frac{2n+1}{4\pi}\cdot\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{im\varphi}.$$

Here, $P_n^{|m|}(x)$ is the associated Legendre polynomial with indices $n = 0, 1, \cdots$ and $|m| \leq n$. Then we have

$$\mathcal{K}_{\partial\Omega}^*[Y_n^m](x) = \frac{1}{2(2n+1)}Y_n^m(\hat{x}),$$
$$|x| = r_0, n = 0, 1, \cdots, m = -n, -n+1, \cdots, n-1, n.$$

$\{Y_n^m(\hat{x});\ n = 1, 2, \cdots, m = -n, -n+1, \cdots, n-1, n\}$ is complete in $L_0^2(\Omega)$, hence in $\mathcal{H}_0^*$, which means that $\mathcal{K}_{\partial\Omega}^*$ has the following eigenfunction expansion:

$$\mathcal{K}_{\partial\Omega}^* = \sum_{n=1}^{\infty}\frac{1}{2(2n+1)}\sum_{m=-n}^{n}\psi_n^m \otimes \psi_n^m,$$

where

$$\psi_n^m(x) = \sqrt{\frac{r_0^3}{2n+1}}Y_n^m(\hat{x}), \quad |x| = r_0.$$

Note that $\{\psi_n^m(x);\ |x| = r_0, n = 1, 2, \cdots, m = -n-n+1, \cdots, n-1, n\}$ is an orthonormal basis in $\mathcal{H}_0^*$. We also have

$$\mathcal{S}_{\partial\Omega}[Y_n^m](x) = \begin{cases} -\dfrac{1}{2n+1}\dfrac{r^n}{r_0^{n-1}}Y_n^m(\hat{x}), & \text{for } |x| = r \leq r_0, \\ -\dfrac{1}{2n+1}\dfrac{r_0^{n+2}}{r^{n+1}}Y_n^m(\hat{x}), & \text{for } |x| = r > r_0, \end{cases}$$

for $n = 0, 1, \cdots$, $m = -n, -n+1, \cdots, n-1, n$. See [2].

Theorem 4.3. *Suppose that Ω is a three dimensional ball. Then there is a constant $C > 0$ such that*

$$\|\nabla u_\delta\|_{L^2(\Omega)} \le C. \tag{38}$$

Proof. By the symmetry, we can assume that $a = (0, 0, a_3)$. Therefore, we only have to study the asymptotics of the following summation:

$$\sum_{j=1}^{\infty} \frac{|a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_j](z)|^2}{\delta^2 + \lambda_j^2} = \sum_{n=1}^{\infty} \frac{a_3^2}{\delta^2 + \frac{1}{4}\left(\frac{1}{2n+1}\right)^2} \cdot \frac{r_0^3}{2n+1} \cdot \left| a_3 \cdot \left(-\frac{1}{2n+1} \frac{\partial}{\partial x_3} \left(\frac{r_0^{n+2}}{r^{n+1}} Y_n^m(\hat{z}) \right) \right) \right|^2,$$

which turns out that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{a_3^2 r_0^3}{\delta^2 + \frac{1}{4}\left(\frac{1}{2n+1}\right)^2} \cdot \left(\frac{1}{2n+1}\right)^3 \left(\frac{r_0}{r}\right)^{n+2} \sum_{m=-n}^{n} |-(n+1)\cos\varphi - im\sin\theta|^2 \, |Y_n^m(\hat{z})|^2 \\
\le &\sum_{n=1}^{\infty} \frac{a_3^2 r_0^3}{\delta^2 + \frac{1}{4}\left(\frac{1}{2n+1}\right)^2} \cdot \left(\frac{1}{2n+1}\right)^3 \cdot \left(\frac{r_0}{r}\right)^{n+2} \cdot 2(n+1)^2 \sum_{m=-n}^{n} |Y_n^m(\hat{z})|^2 .
\end{aligned} \tag{39}$$

By the Unsöld's theorem

$$\sum_{m=-n}^{n} |Y_n^m(\hat{x})|^2 = \frac{2n+1}{4\pi}, \quad n = 0, 1, 2, \cdots,$$

the right hand side of (39) equals

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{a_3^2 r_0^3}{\delta^2 + \frac{1}{4}\left(\frac{1}{2n+1}\right)^2} \cdot \left(\frac{1}{2n+1}\right)^3 \cdot \left(\frac{r_0}{r}\right)^{n+2} \cdot 2(n+1)^2 \cdot \frac{2n+1}{4\pi} \\
= &\frac{2a_3^2 r_0^3}{4\pi} \sum_{n=1}^{\infty} \frac{1}{\delta^2 + \frac{1}{4}\left(\frac{1}{2n+1}\right)^2} \cdot \left(\frac{n+1}{2n+1}\right)^2 \cdot \left(\frac{r_0}{r}\right)^{n+2} \\
\le &\frac{2a_3^2 r_0^3}{\pi} \sum_{n=1}^{\infty} (n+1)^2 \left(\frac{r_0}{r}\right)^{n+2} \\
= &\frac{2a_3^2 r_0^3}{\pi} \cdot \frac{r_0 + r}{(r - r_0)^3},
\end{aligned}$$

which proves the theorem. □

Theorem 4.3 implies that ALR does not occur on ball in three dimensions.

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Department of Electrical and Electronic Engineering and Computer Science
Graduate School of Science and Engineering
Ehime University
3, Bunkyo-cho, Matsuyama, 790-8577
JAPAN
E-mail address: `ando@cs.ehime-u.ac.jp`

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